数理解析研究所講究録
第 1665 巻 2009 年 160-164

On some systems of linear inequalities derived from algebraic groups over number fields
(数体上の代数群に由来する或る種の線型不等式系について)

FUJIMORI, MASAMI (藤森雅巳)
KANAGAWA INSTITUTE OF TECHNOLOGY (神奈川工科大学 基礎・教養教育センター)

14:50–15:20, October 19, 2007

Three years ago, in this same series of workshops, the speaker announced that any anisotropic torus over any number field is obtained as a quotient group of the defining affine group scheme of a neutral Tannakian category of equivalence classes of semi-stable systems of linear inequalities of slope zero. Recently, he has extended the result to arbitrary algebraic groups over an arbitrary number field which is densely generated by tori, whether it is anisotropic or not.
In this talk, the speaker explains his result and its limitations through a few examples of systems of linear inequalities.

$\mathcal{C}$: a cat. of general Roth systems of slope 0

'Objects' of $\mathcal{C}$

Example 1
x, y: indeterminates; $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}} \setminus \mathbb{Q}$;
$f_0, f_1 \in \mathbb{Q}(\alpha)[x, y]$

$$f_0 = x,\ f_1 = \alpha x - y;$$

$c(0), c(1) \in \mathbb{R}$

$$c(0) = -1,\ c(1) = 1.$$

$(f_0, f_1; c(0), c(1))$ *a gen. Roth sys. of slope* 0

Fixed $\delta \in \mathbb{R}_{>0}$; variable $Q \in \mathbb{R}_{>1}$

$$(1) \qquad |x| < Q^{-c(0)-\delta},\ |\alpha x - y| < Q^{-c(1)-\delta} \qquad (x, y \in \mathbb{Z})$$

\# of sol. to (1) is finite (Roth's thm).

In these several years, I'm interested in a category $\mathcal{C}$ of certain equivalence classes of general Roth systems of slope 0. ... In place of the precise definition of the category $\mathcal{C}$, let's look at a few examples of 'objects' of that category.

In the 1st example, we denote by x and y two indeterminates. Take up a real algebraic number α which is irrational. We define linear forms f_0 and f_1 in x and y with coefficients in the field generated by α over the rationals respectively as x and $\alpha x - y$. Real numbers $c(0)$ and $c(1)$ are set to equal -1 and 1, respectively. The data $(f_0, f_1; c(0), c(1))$ is an example of general Roth system of slope 0.

With the help of an arbitrarily fixed positive number δ and ..., we associate the general Roth system with the following linear inequalities. Here, assume the indeterminates x and y take values in the ring of rational integers. In this setting, the number of solutions to the inequalities (1) is finite thanks to the famous Roth's theorem.

Const. $C \in \mathbb{R}_{>0}$

$$(2) \qquad |x| < C \cdot Q^{-c(0)},\ |\alpha x - y| < C \cdot Q^{-c(1)} \qquad (x, y \in \mathbb{Z})$$

$C \gg 0 \Rightarrow$
\# of sol. to (2) is ∞ (Minkowski's thm).

We consider at the same time, for a positive constant C, the next related linear inequalities. The differences from the above inequalities are in the coefficients and in the exponents of the variable real number Q. The area in the xy-plane of the parallelogram defined by the inequalities (2) is a constant times the square of C for any value of the variable Q. So, if the constant C is large enough, then the number of solutions to (2) is infinite by Minkowski's theorem of the geometry of numbers.

Example 2

x, y, z: indeterminates; $\alpha \in \mathbb{R} \cap \overline{\mathbb{Q}}$, $\deg \alpha > 2$;
$f_0, f_1, f_2 \in \mathbb{Q}(\alpha)[x, y, z]$

$$f_0 = x,\ f_1 = \alpha x - y,\ f_2 = \alpha^2 x - z;$$

$c(0), c(1), c(2) \in \mathbb{R}$

$$c(0) = -2,\ c(1) = 1,\ c(2) = 1.$$

$(f_0, f_1, f_2; c(0), c(1), c(2))$
a gen. Roth sys. of slope 0

Fixed $\delta \in \mathbb{R}_{>0}$; variable $Q \in \mathbb{R}_{>1}$

$$(3) \qquad |x| < Q^{-c(0)-\delta},\ |\alpha x - y| < Q^{-c(1)-\delta},\ |\alpha^2 x - z| < Q^{-c(2)-\delta} \qquad (x, y, z \in \mathbb{Z})$$

\# of sol. to (3) is finite (subsp. thm of Schmidt).

Const. $C \in \mathbb{R}_{>0}$

$$(4) \qquad |x| < C \cdot Q^{-c(0)},\ |\alpha x - y| < C \cdot Q^{-c(1)},\ |\alpha^2 x - z| < C \cdot Q^{-c(2)} \qquad (x, y, z \in \mathbb{Z})$$

$C \gg 0 \Rightarrow$
\# of sol. to (4) is ∞ (Minkowski's thm).

In the 2nd example, we denote by x, y, z three indeterminates. For a real algebraic number α of degree larger than 2, we define linear forms f_0, f_1, f_2 with coefficients in $\mathbb{Q}(\alpha)$ in x, y, z respectively as x, $\alpha x - y$, and $\alpha^2 x - z$. Real numbers $c(0), c(1), c(2)$ are set to be equal to $-2, 1, 1$, respectively. The data $(f_0, f_1, f_2; c(0), c(1), c(2))$ is a general Roth system of slope 0.

As in the 1st example, we associate the system with the following linear inequalities. Compared with the inequalities in the 1st example, the indeterminate z and the third inequality are added. The number of solutions in the ring of rational integers to the inequalities (3) is known to be finite by the subspace theorem of Schmidt.

Also as in the 1st example, consider the next related linear inequalities. The volume in the xyz-space of the parallelepiped it defines is a constant times the cube of C. We see that when C is sufficiently large, the number of solutions to the inequalities (4) is infinite by Minkowski's theorem.

Example 3
x: indeterminate; $f, g \in \mathbb{Q}[x]$

$$f = x, \ g = x;$$

$c, d \in \mathbb{R}$

$$c = 1, \ d = -1.$$

$((f; c), (g; d))$ *a gen.* `Roth` *sys. of slope* 0

$p \in \mathbb{Z}$: prime; $|\cdot|_p$: p-adic val., $|p|_p = \dfrac{1}{p}$

$$(5) \quad |x| < Q^{-c-\delta}, \ |x|_p \le Q^{-d} \quad \left(x \in \mathbb{Z}\left[\frac{1}{p}\right]\right)$$

The only sol. to (5) is $x = 0$ (prod. formula).

Const. $C \in \mathbb{R}_{>0}$

$$(6) \quad |x| < C \cdot Q^{-c}, \ |x|_p \le Q^{-d} \quad \left(x \in \mathbb{Z}\left[\frac{1}{p}\right]\right)$$

$C > 1 \Rightarrow$ # of sol. to (6) is ∞.

In the 3rd example, we define rational linear forms f and g in one indeterminate x both as x itself. Real numbers c and d are now set to eqaul 1 and -1, respectively. The data $((f; c), (g; d))$ is also an example of general `Roth` system of slope 0.

Choose a rational prime p. We use the subscript p to denote the standard p-adic absolute value.

Associate the above system with the following inequalities. This time, the second inequality is defining a lattice in the x-line for each value of the variable Q. For this reason, the inequality contains the equality, we do not attach the term $-\delta$ to the exponent of Q, and the indeterminate x is assumed to take values in the ring of rational numbers whose denominators are powers of p. The only solution to (5) is 0 because of the so-called product formula.

For a positive constant C, consider the next related linear inequalities. It's easily seen that if C is larger than 1, then the number of solutions to the inequalities (6) is infinite.

Recent result

k: base field of $\mathcal{C}$, fin./$\mathbb{Q}$; $\widetilde{G}$: affine gp scheme/k

$$\mathcal{C} \simeq \underline{\mathrm{Rep}}_k(\widetilde{G})$$

Thm
Any alg. gp G/k generated by tori
(e.g. any torus or any reductive gp)

$$G \twoheadleftarrow \widetilde{G}: \textit{quotient}$$

i.e.,

$$\underline{\mathrm{Rep}}_k(G) \hookrightarrow \mathcal{C}: \textit{full subcat.}$$

Now let's return to a category $\mathcal{C}$ of general `Roth` system of slope 0. We denote by k its base field which is a finite extension of $\mathbb{Q}$. It's known that there exists an affine group scheme $\widetilde{G}$ defined over k such that the category $\mathcal{C}$ is equivalent to the category of finite dimensional rational representations over k of $\widetilde{G}$.

Our recent result says that any algebraic group G defined over k which is densely generated by subtori, whether it's anisotropic or not, is a quotient group of $\widetilde{G}$. In other words, the category of finite dimensional representations over k of G can be viewed as a full subcategory of $\mathcal{C}$.

Consequence & limitations

Example 1 (conti.)

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ of deg. 2

$$q(x,y) = \mathrm{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}(\alpha x - y)$$

$(f_0, f_1; c(0), c(1)) \leftarrow$ *a rep. of*

$$S = \mathrm{Spec}\,(\mathbb{Q}[x,y]/\,(q(x,y)-1))$$

$m \in \mathbb{Q}$

$$S_m = \mathrm{Spec}\,(\mathbb{Q}[x,y]/\,(q(x,y)-m))$$

$$\{\text{sol. to (2)}\} \subset \coprod_m S_m(\mathbb{Z}) \quad \text{(fin. union)}$$

$Q \gg 0 \Rightarrow$

$$\{\text{sol. to (1)}\} \subset S_0(\mathbb{Z})$$

hence $x = y = 0$
(lin. indep. of 1 and α over $\mathbb{Q}$; Liouville)

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ of deg. > 2,
$(f_0, f_1; c(0), c(1)) \leftarrow$ a rep. of what?

Example 2 (conti.)

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ Galois, of deg. 3;
$r, s, t \in \mathbb{Q}$ satisfying $\alpha^3 + r\alpha^2 + s\alpha + t = 0$

$$n(x,y,z) = \mathrm{Norm}_{\mathbb{Q}(\alpha)/\mathbb{Q}}\left(\frac{t}{\alpha}x - (r+\alpha)y - z\right)$$

$(f_0, f_1, f_2; c(0), c(1), c(2)) \leftarrow$ *a rep. of*

$$T = \mathrm{Spec}\,(\mathbb{Q}[x,y,z]/\,(n(x,y,z)-1))$$

Instead of giving an outline of the proof, ... We have only a partial knowledge in our present situation about the whole group scheme $\widetilde{G}$, though we have found a large class of algebraic groups as quotients of $\widetilde{G}$.

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a quadratic extension, ... The general Roth system $(f_0, f_1; c(0), c(1))$ can be derived from a representation of the next 1-dimensional anisotropic torus S.

For each rational number m, we consider a principal homogeneous space S_m of S of the following type. The solutions to linear inequalities (2) are integer points on a finite number of S_m. The inclusion can be interpreted as a kind of parametrization of an infinite set of the solutions to (2) by integral points on a finite number of principal homogeneous spaces S_m.

For large values of the variable Q, the solutions to inequalities (1) are integer points on S_0. Hence both x and y must be 0, by linear independence of the numbers 1 and α over $\mathbb{Q}$. This is a restatement of a classical theorem of Liouville.

When the extension degree is bigger than 2, the case we do need Roth's theorem to claim the finiteness of the number of solutions to (1), we do not know yet the way to understand the general Roth system $(f_0, f_1; c(0), c(1))$ as coming from a representation of a group defined over $\mathbb{Q}$.

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ is a Galois extension of degree 3, we denote by r, s, t the respective rational coefficients of the monic minimal polynomial of α over $\mathbb{Q}$. We define a cubic polynomial $n(x,y,z)$ as norm of the linear form $\frac{t}{\alpha}x - (r+\alpha)y - z$. The general Roth system $(f_0, f_1, f_2; c(0), c(1), c(2))$ can be derived from a representation of the next 2-dimensional anisotropic torus T.

$m \in \mathbb{Q}$

$$T_m = \operatorname{Spec}\left(\mathbb{Q}[x,y,z]/\left(n(x,y,z)-m\right)\right)$$

$$\{\text{sol. to (4)}\} \subset \coprod_m T_m(\mathbb{Z}) \quad \text{(fin. union)}$$

$Q \gg 0 \Rightarrow$

$$\{\text{sol. to (3)}\} \subset T_0(\mathbb{Z})$$

hence $x = y = z = 0$
(lin. indep. of $1, \alpha, \alpha^2$ over $\mathbb{Q}$)

When $\mathbb{Q}(\alpha)/\mathbb{Q}$ of deg. > 3,
$(f_0, f_1, f_2; c(0), c(1), c(2)) \leftarrow$ a rep. of what?

For rational numbers m, consider principal homogeneous spaces T_m of T of the following type. The solutions to linear inequalities (4) are integer points on a finite number of T_m. As in the previous example, the inclusion can be read as a kind of parametrization of an infinite set of the solutions to (4) by integral points on a finite number of principal homogeneous spaces T_m.

For large values of the variable Q, the solutions to (3) are integer points on T_0. Hence all x, y, z are 0, by linear independence of the numbers $1, \alpha, \alpha^2$ over $\mathbb{Q}$. Note in particular that we do not require Schmidt's subspace theorem in this case to insist on the finiteness of the number of solutions to inequalities (3).

When the degree of extension is bigger than 3, we do not know yet how to regard the general Roth system $(f_0, f_1, f_2; c(0), c(1), c(2))$ as coming from a representation of a group defined over $\mathbb{Q}$. We remark that the subspace theorem of Schmidt assures the finiteness of the number of solutions to (3) also in this latter case.

Example 3 (conti.)
$((f;c),(g;d)) \leftarrow$ *rep. of* $\mathbb{G}_m$

We end this talk, mentioning that the general Roth system $((f;c),(g;d))$ can be derived from the standard 1-dimensional representation of a 1-dimensional split torus defined over $\mathbb{Q}$.

References

M. Fujimori. On representation of algebraic groups over a number field that are generated by tori. Preprint, 10 pp., March 2007.

—. Anisotropic algebraic groups over a number field and a category of systems of linear inequalities. Preprint, 17 pp., November 2006.

—. On systems of linear inequalities. *Bull. Soc. Math. France*, 131(1):41–57, 2003. Corrigenda. *ibid.*, 132(4):613–616, 2004.